

Asymptotic Integral Kernel for Ensembles of Random Normal Matrix with Radial Potentials

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Abstract

We use the steepest descents method to study the integral kernel of a family of normal random matrix ensembles with eigenvalue distribution

$$P_N(z_1, \dots, z_N) = Z_N^{-1} e^{-N \sum_{i=1}^N V_\alpha(z_i)} \prod_{1 \leq i < j \leq N} |z_i - z_j|^2$$

where $V_\alpha(z) = |z|^\alpha$, $z \in \mathbb{C}$ and $\alpha \in]0, \infty[$. Asymptotic analysis with error estimates are obtained. A corollary of this expansion is a scaling limit for the n -point function in terms of the integral kernel for the classical Segal–Bargmann space.

Keywords: Random normal matrices, Integral kernels, Steepest descents method, scaling limit of n -point correlations

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1 Introduction and statement of the main result

The investigation of non-Hermitian random matrices, whose elements are independent complex Gaussian variables without any constraint, began with the work of Ginibre [10]. Applying the theory of Haar measure to the group $GL(N, \mathbb{C})$ of $N \times N$ complex matrices, the joint probability distribution of the eigenvalues has shown to be given by (1.2) with $V(z) = |z|^2$ and the eigenvalue density in the complex plane, defined by

$$\int_A \rho_N(z) d^2 z = \frac{1}{N} \mathbb{E} (\# \{ \text{eigenvalues in } A \})$$

for any Borel set $A \subset \mathbb{C}$, where $\mathbb{E}(\cdot)$ is the expectation with respect to P_N , has shown to converges to the so called circular law

$$\rho(z) = \begin{cases} \frac{1}{\pi} & \text{if } |z| \leq 1 \\ 0 & \text{otherwise} \end{cases} . \quad (1.1)$$

Chau and Yue [4] have subsequently introduced ensembles of random normal matrices in the context of the quantum Hall problem of N electrons in a strong magnetic field, opening a new front of research in the area of random matrices. Since normal matrices are unitarily equivalent to a diagonal matrix, the probability distribution of eigenvalues for random normal ensembles can be achieved, exactly as in the Hermitian ensembles, by choosing an appropriated coordinate system that factorizes the eigenvalues contribution from the rest (see, respectively, Section 5.3 of [6] and [5] for the Hermitian and normal ensembles).

Normal ensembles differ from the Hermitian counterpart by the statistical dependence of matrix elements even for Gaussian ensembles and, most importantly, by the fact that their eigenvalues are generically complex. Among the usual questions concerning the statistics of their eigenvalues there are some related with universality that remain unresolved for the normal ensembles. According to the theory of random matrices, the eigenvalue correlations in Hermitian, and normal ensembles as well, are given by the determinant of an integral kernel whose asymptotic behavior for large N governs their decay. The limit integral kernel is well known to be universal for standard models of Hermitian ensembles (see [6] and references therein). The scenery for normal ensembles, despite of certain efforts in this direction, remains undisclosed.

The present work addresses the integral kernel of ensembles of normal matrices weighed by e^{-NV} with V depending only on the absolute value of eigenvalues. We apply the steepest descents method to obtain scaling limits for the integral kernel with error estimates in power of $1/N$. Our results can be extended for a large class of radial symmetric potentials V satisfying condition (1.3) but we shall restrict ourselves to a sub class of potentials (1.7), for simplicity. Although Chau and Zaboronsky [5] have given asymptotic expressions for one and two-point correlation functions, the integral kernel of normal random matrices has not been previously considered for the models addressed here.

The eigenvalue probability distribution of the ensemble of random normal matrices is given by

$$P_N(z_1, \dots, z_N) = Z_N^{-1} e^{-N \sum_{i=1}^N V(z_i)} \prod_{1 \leq i < j \leq N} |z_i - z_j|^2 \quad (1.2)$$

with potentials $V : \mathbb{C} \rightarrow \mathbb{R}$ satisfying the properties: (i) V is continuous and (ii)

$$\lim_{|z| \rightarrow \infty} \left(\frac{V(z)}{2} - \log z \right) = \infty \quad (1.3)$$

to avoid the eigenvalues escape to infinity (see e.g. Saff and Totik [17]). Equation (1.2) can be written as

$$P_N(z_1, \dots, z_N) = \frac{1}{N!} \det (K_N(z_i, z_j))_{i,j=1}^N \quad (1.4)$$

with K_N being the integral kernel

$$K_N(z, w) = e^{-\frac{N}{2}V(z)} e^{-\frac{N}{2}\overline{V(w)}} \sum_{j=1}^N \phi_j(z) \overline{\phi_j(w)} \quad (1.5)$$

where $\{\phi_j\}_{j=1}^N$ is the set of the orthonormal polynomials with respect to the inner product $(\cdot, \cdot)_{\nu_N}$ with weight

$$d\nu_N(z) = e^{-NV(z)} d^2z$$

and the n -point correlation function associated to P_N can be written as

$$R_n^N(z_1, \dots, z_n) = \det (K_N(z_i, z_j))_{i,j=1}^n. \quad (1.6)$$

The statistics of the eigenvalues are thus given by the asymptotic behavior of the integral kernel.

The main result of this paper is as follows.

Theorem 1.1 *Let*

$$V_\alpha(z) = |z|^\alpha, \quad \alpha > 0 \quad (1.7)$$

be a family of radially symmetric potentials,

$$S(\tau, K) = \left\{ \zeta \in \mathbb{C} : 0 < |\zeta| < K, \quad |\arg \zeta| < \frac{\tau}{2} \right\}$$

be a sectorial domain of opening τ and radius K and, for each $0 < \delta < 1$, let $\gamma = \gamma(\alpha, \delta)$ be such that

$$\alpha\gamma + \delta = 1.$$

Then, the integral kernel (1.5) with $V = V_\alpha$ satisfies

$$\frac{1}{N^{\delta+2\gamma}} K_N^\alpha \left(\frac{Z}{N^\gamma}, \frac{W}{N^\gamma} \right) = \frac{\alpha^2}{4\pi} (Z\bar{W})^{\frac{\alpha}{2}-1} e^{N^\delta \left((Z\bar{W})^{\frac{\alpha}{2}} - \frac{|Z|^\alpha}{2} - \frac{|W|^\alpha}{2} \right)} \left(1 + E_N^{\alpha,\delta}(Z\bar{W}) \right) \quad (1.8)$$

with error estimation

$$\left| E_N^{\alpha,\delta}(\zeta) \right| \leq O(N^{-\delta/2}) \quad (1.9)$$

whenever $\zeta \in S\left(\theta/\sqrt{N}, (2/\alpha)^{2/\alpha} N^{2(1-\delta)/\alpha}\right)$, for some $\theta > 0$ and large enough N .

In particular, taking $\delta \nearrow 1$ and, consequently, $\gamma \searrow 0$ we obtain

$$\frac{1}{N} K_N^\alpha(Z, W) = \frac{\alpha^2}{4\pi} (Z\bar{W})^{\frac{\alpha}{2}-1} e^{N \left((Z\bar{W})^{\frac{\alpha}{2}} - \frac{|Z|^\alpha}{2} - \frac{|W|^\alpha}{2} \right)} \left(1 + E_N^{\alpha,1}(Z\bar{W}) \right) \quad (1.10)$$

with $O\left(1/\sqrt{N}\right)$ error for $Z\bar{W} \in S\left(\theta/\sqrt{N}, (2/\alpha)^{2/\alpha}\right)$.

Remark 1.2 The parameter $\delta < 1$ has been introduced to ensure that the eigenvalues are "sampling" in the bulk, out of any fixed compact domain containing the origin. The case of interest for applications is the limit point $\delta = 1$. The limit, as N goes to infinity, of any function involving the asymptotic expression (1.10) is called **bulk scaling limit** of that function.

Remark 1.3 The restriction to a sector $S(\tau, K)$ of opening τ that shrinks with $1/\sqrt{N}$ is an artifact of our method. Equation (1.8) is expected to hold for $Z\bar{W} \in S(\tau, K)$, with $K = K(\tau) > 0$ for $0 \leq \tau < 4\pi/\alpha$, but our estimates on the error for replacing a sum by an integral, giving by the Euler–Maclaurin sum formula, break down except for sectors $S(\theta N^{-\beta}, K)$ with $\theta > 0$ and $\beta \geq 1/2$ (see (4.33) and following equations). Numerical calculations performed in [21] for $\alpha \geq 2$ indicate that (1.8) might hold for $Z\bar{W} \in S(4\pi/\alpha, K)$ with an error decaying faster than any power of N for some $K < 1$ (see also the next remark for an improved and simple estimate for $\alpha = 2$). There, a different error:

$$\sup_{|z|, |w| < (2/\alpha)^{1/\alpha}; |\arg(z\bar{w})| < 2\pi/\alpha} \left| \frac{\alpha^2}{4\pi} (Z\bar{W})^{\frac{\alpha}{2}-1} e^{N^\delta \left((Z\bar{W})^{\frac{\alpha}{2}} - \frac{|Z|^\alpha}{2} - \frac{|W|^\alpha}{2} \right)} E_N^{\alpha,1}(z\bar{w}) \right|$$

denoted by R_n^α , has been considered. We warn that the result of Theorem 1.1 has been imprecisely stated in Eq. (9) of [21].

Remark 1.4 Taylor remainder formula can be used to estimate the difference between the Taylor polynomial S_N and the function f_N , respectively defined by (3.3) with $\delta = 1$ (see also (4.4)) and by the infinite sum with the same summand. For $\alpha = 2$, $f_N(\zeta) = N\zeta e^{N\zeta}/\pi$. By (3.2), together with the Lagrange remainder, one gets (1.10) with the error function satisfying $|E_N(\zeta)| = O(N^{-1/2}(e|\zeta|)^N e^{-N(1-a)\Re\zeta})$, for some $0 < a < 1$ and large enough N (see calculations in Appendix A). We observe that (1.10) with $\alpha = 2$ holds with $\sup_{\zeta \in \bar{S}(\tau, K)} |E_N(\zeta)| = O(1/\sqrt{N})$ for $\zeta = Z\bar{W}$ in a sectorial domain $S(\tau, K)$ with $K = K(\tau, a) > 0$ given by smallest solution of $Ke^{-(1-a)K \cos \tau/2+1} = 1$.¹ Taylor remainder method, together (perhaps) with some additional ingredient, may be extended for $\alpha > 2$ but it doesn't seems to work for $0 < \alpha < 2$.

Remark 1.5 The asymptotic behavior (1.10) for $\alpha = 2$, without error estimation, was established in [9]. Whether n -point functions are universal for normal ensembles with weight e^{-NV} , where $V(z)$ is a polynomial in $|z|^2$ of positive degree with nonnegative coefficients, was addressed in [5].

It follows from equations (1.10) and (1.6) that normal ensembles with the class of potentials V_α are universal alike the Hermitian ensembles (see e.g. Subsection 5.6.1 of [6] and [13], for recent results):

Corollary 1.6 Let r, z_1, \dots, z_n be $n+1$ complex numbers and write²

$$Z_i = r + \frac{z_i}{\sqrt{\pi K_N^\alpha(r, r)}}. \quad (1.11)$$

¹We thank an anonymous referee for suggesting the use of the Taylor remainder to estimate the error in (1.10) for $\alpha = 2$.

²For Hermitian ensembles, $z_i/\sqrt{\pi K_N^\alpha(r, r)}$ and the universal integral kernel $\mathbb{K}(z, w)$ in (1.12) are respectively replaced by $x_i/(\pi K_N^\alpha(r, r))$ and by the Sinc function $\mathbb{S}(x-y) = \frac{\sin(x-y)}{\pi(x-y)}$.

Then, the following scaling limit for the n -point function

$$\lim_{N \rightarrow \infty} \frac{1}{\pi^n K_N^\alpha(r, r)^n} R_n^N(Z_1, \dots, Z_n) = \det(\mathbb{K}(z_i, z_j))_{i,j=1}^n \quad (1.12)$$

holds uniformly for r in any compact set of the open set $\{z \in \mathbb{C} : 0 < |z| < (2/\alpha)^{1/\alpha}\}$, where

$$\mathbb{K}(z, w) = \frac{1}{\pi} e^{\left(z\bar{w} - \frac{|z|^2}{2} - \frac{|w|^2}{2}\right)} \quad (1.13)$$

is the integral kernel for the classical Segal–Bargmann space of entire functions. The bulk scaling limit (1.12) is universal in the sense that it is independent of the family of potentials V_a .

We shall address this and other issues related with the conformal invariance of the integral kernel (1.5) in a forthcoming paper [20]. Since the cancellations involved makes the implication of (1.12) far of being straightforward, a complete, although short, proof has been included in Appendix B.

For $n = 2$, (1.12) reads

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{\pi^2 K_N^\alpha(r, r)^2} R_n^N(Z_1, Z_2) &= (\mathbb{K}(z_1, z_1)\mathbb{K}(z_2, z_2) - \mathbb{K}(z_1, z_2)\mathbb{K}(z_2, z_1)) \\ &= \frac{1}{\pi^2} \left(1 - e^{-|z_1 - z_2|^2}\right), \end{aligned} \quad (1.14)$$

a result already obtained for more general radial potentials (see Theorem 1 of [5]). Under the assumption that (1.10) can be extended to the sectorial domain $S(4\pi/\alpha, K)$ (this actually holds for $\alpha = 2$. See Appendix A), a change of variables in the integral Kernel by the function $\varphi_N(z) = (z/\sqrt{N})^{2/\alpha}$, which maps conformally $\{z \in \mathbb{C} : |z| < K^{\alpha/2}\sqrt{N}\}$ into $S(4\pi/\alpha, K)$, yields

$$\lim_{N \rightarrow \infty} \varphi'_N(z) K_N^\alpha(\varphi_N(z), \varphi_N(w)) \overline{\varphi'_N(w)} = \mathbb{K}(z, w) \quad (1.15)$$

where $\mathbb{K}(z, w)$ is the integral kernel given by (1.13). This notion of universality has been called conformal universality in [21]. The estimates in Appendix A establishes the pointwise limit (1.15) in $\mathbb{C} \times \mathbb{C}$ for $\alpha = 2$.

Theorem 1.1 will be proven in Section 4. Sections 2 and 3 contain preliminary materials. The technical part of our result concerns with the error estimation of Euler–Maclaurin formula. Different methods needs to be employed depending on the regions considered in the sum. Appendix A estimates the Taylor remainder of (3.3) for $\delta = 1$ and $\alpha = 2$ and Appendix B proves Corollary 1.6.

2 Ensemble of random normal matrices

We begin with the following

Definition 2.1 *By normal ensembles we mean a probability measure*

$$P(M_N) dM_N = Z_N^{-1} e^{-N \text{Tr} V(M_N)} dM_N \quad (2.1)$$

on the set of $N \times N$ complex matrices M_N supported on the variety $[M_N, M_N^*] = 0$ and invariant by unitary conjugation $\tilde{M}_N = U_N^* M_N U_N$:

$$P(M_N) dM_N = P(\tilde{M}_N) d\tilde{M}_N. \quad (2.2)$$

The elements $m_{ij} = m_{ij}^R + im_{ij}^I$, $1 \leq i \leq j \leq N$ of M_N in the normal ensemble cannot be picked independently according to any product measure, absolutely continuous with respect to the Lebesgue measure $\prod_{1 \leq i \leq j \leq N} dm_{ij}^R dm_{ij}^I$ in \mathbb{R}^{N^2+N} , even when the weight $e^{-N \text{Tr} V(M_N)}$ is Gaussian, in view of the constraint on elements m_{ij} with $i > j$ ³. So, the elements of M_N when sampling on normal ensembles are always statistically dependent. Note that the set of normal matrices with simple spectra is open and dense in \mathbb{R}^{N^2+N} and has full measure (see [6] for a proof in the Hermitian ensembles).

As M_N is normal, M_N is unitarily equivalent to a diagonal matrix of eigenvalues and there exist U_N satisfying $U_N^{-1} = U_N^*$ and

$$M_N = U_N \Lambda_N U_N^* \quad (2.3)$$

with $\Lambda_N = \text{diag}\{z_1, \dots, z_N\}$, ordered according their absolute value: $|z_i| \leq |z_j|$ if $i < j$. Following section 5.3 of [6] with few adjustments (see [5] and [8]), the spectral decomposition (2.3) considered as a change of variables $M_N \xrightarrow{\varphi} (\Lambda_N, U_N \bmod \mathbb{T}^N)$ yields

$$P(M_N) dM_N = Z^{-1} e^{-N \sum_{i=1}^N V(z_i)} J(z, p) \prod_{1 \leq i \leq N} d^2 z_i \prod_{1 \leq j \leq l} d^2 p_k, \quad (2.4)$$

where $\{p_i\}_{i=1}^l$ with $2l + N = N^2$, are variables associated with the eigenvectors of M , $d^2 z$ denotes the Lebesgue measure in \mathbb{C} and

$$J(z, p) = \prod_{1 \leq i < j \leq N} |z_i - z_j|^2 f(p)$$

is the Jacobian of φ , with f a function depending only on the eigenvectors variables $\{p_i\}_{i=1}^l$. The eigenvalue probability distribution (2.1) of this ensemble is obtained integrating (2.4) with respect to $\{p_i\}_{i=1}^l$.

The n -point correlation function is defined by (see e.g. [16])

$$R_n^N(z_1, \dots, z_n) = \frac{N!}{(N-n)!} \int P_N(z_1, \dots, z_N) \prod_{i=n+1}^N d^2 z_i \quad (2.5)$$

and it can be written as (1.6). Stochastic processes of this form are called random determinantal point fields [18]. The present work concerns with the asymptotic analysis of the integral kernel (1.5) and its implications to the limit of the n -point correlation function. We have seen that the limit of the 2-point correlation (1.14) can be read directly from the asymptotic formula (1.10). The eigenvalue density ρ^{V_α} , associated with the normal ensemble defined by V_α , is given by

$$\rho^{V_\alpha}(z) = \lim_{N \rightarrow \infty} \frac{1}{N} R_1^N(z) = \lim_{N \rightarrow \infty} \frac{1}{N} K_N(z, z) = \frac{\alpha^2}{4\pi} |z|^{\alpha-2} \quad (2.6)$$

³If they were independent, it would contradict Schur–Toeplitz statement (see e.g. [12]): “any square matrix is unitarily similar to an upper (or lower) triangular matrix”.

for $|z| \leq (2/\alpha)^{1/\alpha}$ (see Remarks 3.4, for more comment on this). Note that $\rho^{V_a}(z)d^2z$ and the equilibrium or extremal measure $d\hat{\sigma}(z)$ (see e.g. [11]) agree and are supported on the same domain.

3 Integral kernel of normal *ensembles* defined by V_α and various estimates

The present section is devoted to preliminary results on the integral kernel (1.5).

Let $L^2(\mathbb{C}, \nu)$ denote the Hilbert space of square-integrable complex-valued functions

$$\|f\|_\nu^2 = \int_{\mathbb{C}} |f(z)|^2 d\nu(z) < \infty$$

with respect to a positive finite Borel measure ν on \mathbb{C} which, in order to ensure that all analytic polynomials belong to the space is assumed to satisfy

$$\int_{\mathbb{C}} |z|^{2n} d\nu(z) < \infty, \quad n \in \mathbb{N}.$$

If $P_N(\mathbb{C}, \nu)$ denotes the N -dimensional linear vector space of analytic polynomials of degree less than or equal $N - 1$, endowed with the inner product

$$(p, q)_\nu = \int_{\mathbb{C}} \overline{p(z)} q(z) d\nu(z), \quad (3.1)$$

we have

Proposition 3.1 *For each $N \in \mathbb{N}$, the monomials*

$$\phi_j^\alpha(z) = \sqrt{\frac{\alpha}{2\pi\Gamma(2j/\alpha)}} N^{j/\alpha} z^{j-1}$$

with $j = 1, \dots, N$, form an orthonormal set in $P_N(\mathbb{C}, \nu_N^\alpha)$ with respect to

$$d\nu_N^\alpha(z) = e^{-N|z|^\alpha} d^2z, \quad \alpha > 0.$$

The integral kernel (1.5) reads in this case

$$K_N^\alpha(z, w) = e^{-\frac{N}{2}|z|^\alpha} e^{-\frac{N}{2}|w|^\alpha} \tilde{K}_N^\alpha(z, w) \quad (3.2)$$

where

$$\tilde{K}_N^\alpha(z, w) = \frac{\alpha}{2\pi} \sum_{j=1}^N \frac{N^{2j/\alpha} (z\bar{w})^{j-1}}{\Gamma(2j/\alpha)} \quad (3.3)$$

is a reproducing kernel on $P_N(\mathbb{C}, \nu_N^\alpha)$.

Remark 3.2 *For the Bergman space $A^2(\Omega)$ of square-integrable single-valued analytic function on a compact domain Ω , there always exist a complete set of orthonormal polynomials $\{\phi_j(z)\}_{j=1}^\infty$ and the integral kernel*

$$\tilde{K}(z, w) = \sum_{j=1}^\infty \phi_j(z) \overline{\phi_j(w)}$$

converges $\lim_{N \rightarrow \infty} \sum_{j=1}^N \phi_j(z) \overline{\phi_j(w)} = \tilde{K}(z, w)$ uniformly for any z, w in Ω [2]. This is not necessarily the case for unbounded domain but the same properties hold for Segal–Bargmann spaces $A^2(\mathbb{C}; \nu)$ of single-valued analytic functions in \mathbb{C} , square-integrable with respect to $e^{-|z|^2} d^2z$. We call the reader's attention to the N dependence on the inner product (3.1) and the fact that the limit N to infinity in (3.2) involves also a limit of the measure ν_N^α . As one sees from (1.10), together with

$$\frac{|z|^\alpha}{2} + \frac{|w|^\alpha}{2} - \Re(z\bar{w})^{\alpha/2} = \frac{1}{2} |z^{\alpha/2} - w^{\alpha/2}|^2 \geq 0 ,$$

(equality iff $z = w$) and equation (2.6), the limit as $N \rightarrow \infty$ of $K_N^\alpha(z, w)$ goes 0 for $z \neq w$ and diverges for $z = w$.

We shall use (3.3) to obtain an asymptotic expression as stated in Theorem 1.1.

Proof of Proposition 3.1. We need to verify that the monomials are orthogonal with respect to the inner product (3.1). Writing

$$\phi_j(z) = \frac{z^{j-1}}{\sqrt{2\pi I_j}}$$

with $z = re^{i\theta}$, we have

$$\begin{aligned} (\phi_k(z), \phi_j(z))_{\nu_N^\alpha} &= \frac{1}{2\pi \sqrt{I_k I_j}} \int \overline{z^k} z^j e^{-N|z|^\alpha(z)} d^2z \\ &= \frac{1}{\sqrt{I_k I_j}} \int_0^\infty r^{k+j+1} e^{-Nr^\alpha} dr \frac{1}{2\pi} \int_0^{2\pi} e^{i\theta(j-k)} d\theta = \delta_{k,j} , \end{aligned}$$

with the Kroneker delta function $\delta_{k,j} = 1$ if $k = j$ and 0 otherwise, provided

$$I_j = \int_0^\infty r^{2j-1} e^{-Nr^\alpha} dr = \frac{N^{-2j/\alpha}}{\alpha} \Gamma\left(\frac{2j}{\alpha}\right) .$$

Consequently, any analytic polynomial $p(z)$ in $P_N(\mathbb{C}, \nu_N^\alpha)$ can be written as

$$p(z) = \sum_{j=1}^N c_j \phi_j(z) \tag{3.4}$$

with Fourier coefficients

$$c_j = (\phi_j, p)_{\nu_N^\alpha} = \int_{\mathbb{C}} \overline{\phi_j(w)} p(w) e^{-N|w|^\alpha} d^2w . \tag{3.5}$$

Inserting (3.5) into (3.4), gives $p(z) = \left(\tilde{K}_N^\alpha(z, \cdot), p \right)_{\nu_N^\alpha}$ where

$$\tilde{K}_N^\alpha(z, w) = \sum_{j=1}^N \phi_j(z) \overline{\phi_j(w)} = \frac{\alpha}{2\pi} \sum_{j=1}^N \frac{N^{2j/\alpha} (z\bar{w})^{j-1}}{\Gamma(2j/\alpha)} . \tag{3.6}$$

□

Looking for an asymptotic expansion of (3.2), a complex valued function is defined on the positive real line $\mathbb{R}_+ = (0, \infty)$ coinciding with the summand of the integral kernel (3.6) on \mathbb{N} . For fixed numbers $\alpha > 0$, $0 < \delta < 1$, $\zeta \in \mathbb{C} \setminus \{0\}$ and N a positive integer, let $g_\zeta : \mathbb{R}_+ \rightarrow \mathbb{C}$ be given by

$$g_\zeta(x) = \frac{\left(N^{\frac{2\delta}{\alpha}} \zeta\right)^x}{\Gamma(2x/\alpha)} \quad (3.7)$$

and note that $|g_\zeta(x)| = g_{|\zeta|}(x)$.

Lemma 3.3 *Under the above conditions on α , δ , ζ and N , the real valued function $g_{|\zeta|} : \mathbb{R}_+ \rightarrow \mathbb{R}$ has a global maximum*

$$g_{|\zeta|}(x) \leq \max_{x \geq 0} g_{|\zeta|}(x) = g_{|\zeta|}(x^*)$$

at $x^* = x^*(\alpha, \delta, |\zeta|, N) > 0$. For N large enough so that $N > N_0$,

$$N_0 = \max \left(\left(\frac{k}{|\zeta|} \right)^{\frac{\alpha}{2\delta}}, \left(\frac{\alpha}{2} |\zeta|^{\frac{\alpha}{2}} \right)^{\frac{1}{1-\delta}} \right) \quad (3.8)$$

with k a large universal constant, the inequality

$$0 < x^* < N$$

holds and

$$g_{|\zeta|}(x^*) = \frac{1}{\sqrt{2\pi}} |\zeta|^{\frac{\alpha}{4}} N^{\frac{\delta}{2}} \exp \left(|\zeta|^{\frac{\alpha}{2}} N^\delta \right) \left(1 + O \left(\frac{1}{N^\delta} \right) \right) \quad (3.9)$$

$$x^* = \frac{\alpha}{2} |\zeta|^{\frac{\alpha}{2}} N^\delta - \frac{\alpha}{4} + O \left(\frac{1}{N^\delta} \right) \quad (3.10)$$

Proof. Differentiating $g_{|\zeta|}(x)$ with respect to x , we have

$$g'_{|\zeta|}(x) = g_{|\zeta|}(x) \left(\log \left(N^{\frac{2\delta}{\alpha}} |\zeta| \right) - \frac{2}{\alpha} \psi \left(\frac{2}{\alpha} x \right) \right) \quad (3.11)$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function. Since $g_{|\zeta|}(x)$ does not vanish and $\psi(x)$ belongs to a Pick class of functions that can be analytically continued through \mathbb{R}_+ (see e.g. [7]), as x varies in the semi-line $\psi(x)$ increases monotonously from $-\infty$ to ∞ and the maximum of $g_{|\zeta|}$ is attained at the unique solution $x = x^*$ of

$$\log \left(N^{\frac{2\delta}{\alpha}} |\zeta| \right) - \frac{2}{\alpha} \psi \left(\frac{2}{\alpha} x \right) = 0. \quad (3.12)$$

For N so large that the asymptotic expansion [1]

$$\psi(y) \sim \log y + \frac{1}{2y} - \sum_{j=1}^{\infty} B_{2j} \frac{1}{2j y^{2j}} \quad (3.13)$$

of digamma function at $y = N^{\frac{2\delta}{\alpha}} |\zeta|$ can be applied (i. e., $y > k$ where k is the constant mention in (3.8)), we have by (3.12)

$$\log \left(N^\delta |\zeta|^{\frac{\alpha}{2}} \right) = \log \frac{2}{\alpha} x^* + \frac{\alpha}{4x^*} + O \left(\frac{1}{x^{*2}} \right)$$

or equivalently,

$$\frac{\alpha N^\delta |\zeta|^{\frac{\alpha}{2}}}{2} = x^* + \frac{\alpha}{4} + O \left(\frac{1}{x^*} \right)$$

which establishes (3.10). The coefficients B_{2j} in (3.13) are the Bernoulli numbers:

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} .$$

For (3.8), it suffices to solve $\alpha N^\delta |\zeta|^{\frac{\alpha}{2}} / 2 \leq N$ for N . For (3.9), we plug (3.10) into $g_{|\zeta|}(x^*)$. As x^* is order N^δ and, therefore, large enough for applying Stirling formula,

$$\begin{aligned} g_{|\zeta|}(x^*) &= \frac{\left(N^{\frac{2\delta}{\alpha}} |\zeta| \right)^{x^*}}{\Gamma \left(\frac{2}{\alpha} x^* \right)} \\ &= \sqrt{\frac{x^*}{\alpha\pi}} \left(\frac{\alpha e}{2x^*} \right)^{\frac{2}{\alpha} x^*} \left(|\zeta| N^{\frac{2\delta}{\alpha}} \right)^{x^*} \left(1 + O \left(\frac{1}{N^\delta} \right) \right) \\ &= \frac{|\zeta|^{\frac{\alpha}{4}}}{\sqrt{2\pi}} N^{\frac{\delta}{2}} e^{N^\delta |\zeta|^{\frac{\alpha}{2}}} \left(1 + O \left(\frac{1}{N^\delta} \right) \right) . \end{aligned} \quad (3.14)$$

□

Remark 3.4 *Lemma 3.3 still holds for $\delta = 1$ provided $0 < |\zeta| \leq (2/\alpha)^{2/\alpha}$. Note that $x^* = N - \alpha/4 + O(1/N) < N$ for $|\zeta| = (2/\alpha)^{2/\alpha}$, which defines the domain boundary of the density of eigenvalues (2.6) (recall $\zeta = Z\bar{W}$ and $|Z|, |W| \leq (2/\alpha)^{1/\alpha}$).*

The limit $\lim_{N \rightarrow \infty} \tilde{K}_N^\alpha(z, w)/N$ calculated at $z\bar{w} = \zeta/N^{2/\alpha}$, given by the series

$$(\alpha/2\pi) \sum_{j=1}^{\infty} \zeta^{j-1} / \Gamma(2j/\alpha) ,$$

converges uniformly in compact sets of \mathbb{C} to an entire function of ζ of order $\alpha/2$, whose maximum is determined, essentially, by a single term of the series, the so called central index $j^* = j^*(|\zeta|)$. The next result estimates the range of indices j in (3.6) the contributes for its asymptotic expansion for large N .

Lemma 3.5 *Let x be a point that is at least $N^{\frac{\delta}{2}} \log N$ away from the global maximum (3.10) of $g_{|\zeta|}(x)$, that is,*

$$|x - x^*| \geq N^{\frac{\delta}{2}} \log N. \quad (3.15)$$

Then

$$g_{|\zeta|}(x) \leq \max \left(g_{|\zeta|}(x_+), g_{|\zeta|}(x_-) \right) \quad (3.16)$$

where $x_{\pm} = x^ \pm N^{\frac{\delta}{2}} \log N$ and*

$$g_{|\zeta|}(x_{\pm}) = \frac{1}{N^{2 \log N / (\alpha^2 |\zeta|^{\alpha/2})}} g_{|\zeta|}(x^*) \left(1 + O \left(\frac{\log^3 N}{N^{\delta/2}} \right) \right) . \quad (3.17)$$

Proof. (3.16) follows by uniqueness of the maximum value. For (3.17), we repeat the estimates that lead to (3.14) with x_{\pm} in the place of x^* :

$$g_{|\zeta|}(x_{\pm}) = \sqrt{\frac{x_{\pm}}{\alpha\pi}} \left(\frac{e\alpha N^{\delta} |\zeta|^{\frac{\alpha}{2}}}{2x_{\pm}} \right)^{2x_{\pm}/\alpha} \left(1 + O\left(\frac{1}{N^{\delta}}\right) \right). \quad (3.18)$$

Plugging

$$x_{\pm} = \frac{\alpha}{2} |\zeta|^{\frac{\alpha}{2}} N^{\delta} \pm N^{\frac{\delta}{2}} \log N - \frac{\alpha}{4} + O\left(\frac{1}{N^{\delta}}\right)$$

into each term that appears in (3.18), yields

$$\sqrt{\frac{x_{\pm}}{\alpha\pi}} = \sqrt{\frac{N^{\delta} |\zeta|^{\frac{\alpha}{2}}}{2\pi}} \left(1 + O\left(\frac{\log N}{N^{\frac{\delta}{2}}}\right) \right),$$

$$\begin{aligned} \frac{e\alpha N^{\delta} |\zeta|^{\frac{\alpha}{2}}}{2x_{\pm}} &= e \left(1 \pm \frac{2}{\alpha |\zeta|^{\alpha/2}} \frac{\log N}{N^{\delta/2}} - \frac{1}{2 |\zeta|^{\alpha/2}} \frac{1}{N^{\delta}} + O\left(\frac{1}{N^{2\delta}}\right) \right)^{-1} \\ &= \exp \left(1 \mp \frac{2}{\alpha |\zeta|^{\alpha/2}} \frac{\log N}{N^{\delta/2}} + \frac{2}{\alpha^2 |\zeta|^{\alpha}} \frac{\log^2 N}{N^{\delta}} + \frac{1}{2 |\zeta|^{\alpha/2}} \frac{1}{N^{\delta}} + O\left(\frac{\log N}{N^{3\delta/2}}\right) \right), \end{aligned}$$

where we have used

$$\frac{e}{1+\kappa} = \exp(1 - \log(1+\kappa)) = \exp\left(1 - \kappa + \frac{\kappa^2}{2} + O(\kappa^3)\right)$$

and, therefore,

$$\left(\frac{e\alpha N^{\delta} |\zeta|^{\frac{\alpha}{2}}}{2x_{\pm}} \right)^{2x_{\pm}/\alpha} = \exp \left(|\zeta|^{\alpha/2} N^{\delta} - \frac{2}{\alpha^2 |\zeta|^{\alpha/2}} \log^2 N \right) \left(1 + O\left(\frac{\log^3 N}{N^{\delta/2}}\right) \right)$$

Replacing in (3.18), together with (3.9), results (3.17). □

We need one more ingredient.

Lemma 3.6 *Let $f : [a, b] \rightarrow \mathbb{R}$ be a convex function:*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

for any $x, y \in [a, b]$ and $0 < \lambda < 1$ and let

$$P : a = x_0 < \dots < x_K = b$$

be the partition of $[a, b]$ into K equally spacing subintervals of length Δ :

$$x_j = a + j\Delta, \quad j \in \{0, \dots, K\}.$$

Define $t_j \in [x_j, x_{j+1}]$ by the mean value theorem:

$$\int_{x_j}^{x_{j+1}} f(x) dx = f(t_j) \Delta. \quad (3.19)$$

Then, the error in the trapezoidal approximation to the integral

$$\Sigma(f; P) := \sum_{j=0}^{K-1} \left(\int_{x_j}^{x_{j+1}} f(x) dx - \frac{1}{2} (f(x_j) + f(x_{j+1})) \Delta \right) \quad (3.20)$$

is bounded by

$$0 \geq \Sigma(f; P) \geq \left(-\frac{f(t_0)}{2} + \frac{f(x_1)}{2} + \frac{f(x_K)}{2} - \frac{f(t_K)}{2} \right) \Delta. \quad (3.21)$$

Proof. Without loss of generality, we suppose that f is a positive convex function. Let $\{k_j\}_{j=0}^{2K}$ be a numerical sequence defined by

$$\begin{aligned} k_{2j} &= \int_{x_j}^{x_{j+1}} f(x) dx \\ k_{2j+1} &= f(x_{j+1}) \Delta \end{aligned} \quad (3.22)$$

for $j \in \{0, \dots, K-1\}$ and note that, by the mean value theorem (3.19),

$$k_{2j} = f(t_j) \Delta \quad (3.23)$$

for some $t_j \in [x_j, x_{j+1}]$. We shall prove, by a geometric argument together with the convexity of f , that the following inequality

$$k_i \leq \frac{k_{i+1} + k_{i-1}}{2} \quad (3.24)$$

holds for each $i \in \{1, \dots, 2K-1\}$.

Since f is convex, the inequality (3.24) for $i = 2j$:

$$\int_{x_j}^{x_{j+1}} f(x) dx = k_{2j} \leq \frac{k_{2j+1} + k_{2j-1}}{2} = \frac{f(x_{j+1}) + f(x_j)}{2} \Delta$$

is verified comparing the area under the function f in the interval $[x_j, x_{j+1}]$ (left side of (3.24)) with the area of a trapezoid formed by the points $(x_j, 0)$, $(x_{j+1}, 0)$, $(x_j, f(x_j))$ and $(x_{j+1}, f(x_{j+1}))$ (right side of (3.24)).

Once again, by convexity of f , the inequality (3.24) for $i = 2j+1$:

$$f(x_{j+1}) \Delta = k_{2j+1} \leq \frac{k_{2j} + k_{2j+2}}{2} = \frac{1}{2} \int_{x_j}^{x_{j+2}} f(t) dt \quad (3.25)$$

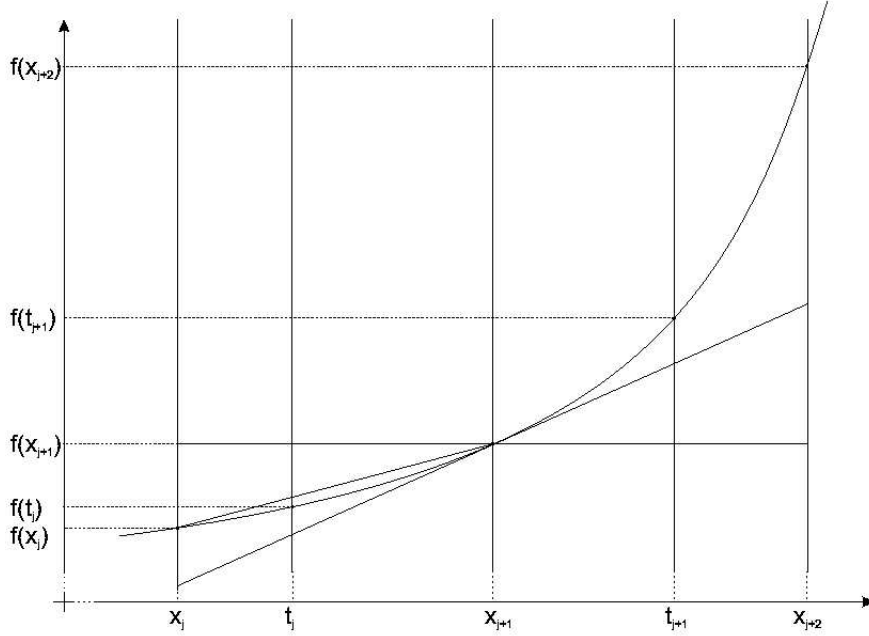
can be verified comparing the area under the function f in the interval $[x_j, x_{j+2}]$ ($2 \times$ the right side of (3.25)) with the area of a rectangle of base in the interval $[x_j, x_{j+2}]$ and height $f(x_{j+1})$ ($2 \times$ the left side of (3.25)).

The later assertion is facilitate if the rectangle is replaced by a trapezoid of same area obtained by rotating the horizontal segment at the top around the point $(x_{j+1}, f(x_{j+1}))$ until it becomes tangent to the graph of f at that point (see figure below).

Now let us consider the sum

$$\begin{aligned}\Sigma_1 &= \sum_{j=0}^{2K} (-1)^j k_j = k_0 - k_1 + \cdots - k_{2K-1} + k_{2K} \\ &= \frac{k_0}{2} - \sum_{j=0}^{K-1} \left(k_{2j+1} - \frac{k_{2j} + k_{2j+2}}{2} \right) + \frac{k_{2K}}{2}\end{aligned}\quad (3.26)$$

$$= k_0 - \frac{k_1}{2} + \sum_{j=1}^{K-1} \left(k_{2j} - \frac{k_{2j-1} + k_{2j+1}}{2} \right) - \frac{k_{2K-1}}{2} + k_{2K}.\quad (3.27)$$



From (3.24) and (3.26), we have

$$\Sigma_1 = \frac{k_0}{2} - \sum_{j=0}^{K-1} \left(k_{2j+1} - \frac{k_{2j} + k_{2j+2}}{2} \right) + \frac{k_{2K}}{2} \geq \frac{k_0}{2} + \frac{k_{2K}}{2}\quad (3.28)$$

and from (3.24) and (3.27), we have

$$\Sigma_1 = k_0 - \frac{k_1}{2} + \sum_{j=1}^{K-1} \left(k_{2j} - \frac{k_{2j-1} + k_{2j+1}}{2} \right) - \frac{k_{2K-1}}{2} + k_{2K} \leq k_0 - \frac{k_1}{2} - \frac{k_{2K-1}}{2} + k_{2K}.\quad (3.29)$$

Since equations (3.20) and (3.27) are related by definition of $\{k_j\}_{j=0}^{2K}$ as

$$\Sigma_1 = k_0 - \frac{k_1}{2} + \Sigma - \frac{k_{2K-1}}{2} + k_{2K}$$

the lower (3.28) and the upper (3.29) bounds yields

$$\frac{k_0}{2} + \frac{k_{2K}}{2} \leq k_0 - \frac{k_1}{2} + \Sigma - \frac{k_{2K-1}}{2} + k_{2K} \leq k_0 - \frac{k_1}{2} - \frac{k_{2K-1}}{2} + k_{2K}$$

or, equivalently,

$$-\frac{k_0}{2} + \frac{k_1}{2} + \frac{k_{2K-1}}{2} - \frac{k_{2K}}{2} \leq \Sigma \leq 0$$

which, in view of definitions (3.22) and (3.23), concludes the proof of lemma. \square

Corollary 3.7 *Let $f : [a, b] \rightarrow \mathbb{R}$ be a concave function:*

$$f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y)$$

for all $x, y \in [a, b]$ and $0 < \lambda < 1$ and let $P, (t_j)_{j=1}^K$ and $\Sigma(f; P)$ be as in the previous lemma. Then

$$0 \leq \Sigma(f; P) \leq \left(-\frac{f(t_0)}{2} + \frac{f(x_1)}{2} + \frac{f(x_K)}{2} - \frac{f(t_K)}{2} \right) \Delta x.$$

\square

Remark 3.8 *The ideas of this proof was based in an argument used to establish the phenomenon of Fresnel diffraction (see e.g. [3]).*

We are now in position to prove Theorem 1.1.

4 Proof of Theorem 1.1

We shall proceed the asymptotic analysis applying the steepest descents method to the integral kernel (3.2). For this we assume N to be large in comparison to all other variables which, from now on, are kept fixed.

It is convenient rewrite z, w and the difference of their argument using scale parameters γ and β :

$$\begin{aligned} Z &= zN^\gamma \\ W &= wN^\gamma, \quad \gamma > 0 \end{aligned} \tag{4.1}$$

and

$$\theta = N^\beta (\arg z - \arg w), \quad \beta > 0 \tag{4.2}$$

The equation (1.5) can thus be written as

$$K_N^\alpha \left(\frac{Z}{N^\gamma}, \frac{W}{N^\gamma} \right) = \frac{\alpha}{2\pi} e^{-N^{1-\alpha\gamma}|Z|^\alpha/2} e^{-N^{1-\alpha\gamma}|W|^\alpha/2} \frac{N^{2\gamma}}{Z\bar{W}} S_N \tag{4.3}$$

where

$$S_N := \sum_{j=1}^N \frac{(N^{2(1-\alpha\gamma)/\alpha} Z\bar{W})^j}{\Gamma(2j/\alpha)}. \tag{4.4}$$

We introduce another auxiliary scale parameter δ satisfying $0 < \delta < 1$ and

$$\alpha\gamma + \delta = 1, \tag{4.5}$$

in order to adjust the spacing in the label that indexes the sum. Note that γ and δ are not independent. Equation (4.4) can be written as

$$S_N = \sum_{j=0}^{N-1} \frac{(N^{2\delta/\alpha} Z \bar{W})^{y_j N^\delta}}{\Gamma(2y_j N^\delta / \alpha)} \quad (4.6)$$

where

$$y_j = N^{-\delta} + jN^{-\delta}, \quad j = 0, \dots, N-1. \quad (4.7)$$

Given a function f of the class $C^{(p)}$ in $[a, b]$, the Euler-Maclaurin sum formula (see e.g. [1] with $\omega = 0$ and $p = 1$)

$$\sum_{j=0}^{N-1} f(y_j) = \frac{1}{h} \int_a^b f(x) dx + R_1 + R_2 \quad (4.8)$$

associated with the uniform partition $P : a = y_0 < y_1 < \dots < y_N = b$,

$$y_j = a + jh,$$

for $j \in \{0, \dots, N-1\}$, can be employed to estimate the errors

$$R_1 = \frac{1}{2} (f(b) - f(a))$$

and

$$R_2 = -h \int_0^1 \left(\frac{1}{2} - t \right) \left(\sum_{j=0}^{N-1} f'(a + (j+t)h) \right) dt$$

in replacing the Darboux–Riemann sum of f by its integral.

We take

$$f(y) = g_{Z\bar{W}}(yN^\delta) = \frac{(N^{2\delta/\alpha} Z \bar{W})^{yN^\delta}}{\Gamma(2yN^\delta/\alpha)}, \quad (4.9)$$

in (4.8) with $g_\zeta(x)$ defined by (3.7). The partition $N^{-\delta} = y_0 < y_1 < \dots < y_{N-1} = N^{1-\delta}$ of $[N^{-\delta}, N^{1-\delta}]$ is chosen with the y_j 's given by (4.7). In order to simplify the notation in (4.9), from now on we fix $\zeta = Z\bar{W} = |\zeta| e^{i\theta/N^\beta}$.

Equation (4.4) can thus be written as

$$S_N = N^\delta \int_{N^{-\delta}}^{N^{1-\delta}} g_\zeta(N^\delta y) dy + r_1 + r_2, \quad (4.10)$$

where

$$r_1 = \frac{1}{2} (g_\zeta(N) - g_\zeta(1)) \quad (4.11)$$

and

$$\begin{aligned} r_2 &= -N^{-\delta} \int_0^1 \left(\frac{1}{2} - t \right) \left(\sum_{j=0}^{N-1} f'(N^{-\delta} + (j+t)N^{-\delta}) \right) dt \\ &= -\sum_{j=0}^{N-1} \int_0^1 \left(\frac{1}{2} - t \right) df((j+t)N^{-\delta}) \\ &= -\sum_{j=0}^{N-1} \int_0^1 \left(\frac{1}{2} - t \right) dg_\zeta(j+t). \end{aligned} \quad (4.12)$$

The proof now proceeds in two parts. The longest one, Part **I**, concerns with the estimates of r_1 and r_2 . Part **II** applies the method of steepest descents to the integral term of the representation (4.8).

I. Estimate of r_1 and r_2 . By the Stirling formula (see (3.14)),

$$g_\zeta(N) = \frac{(N^{2\delta/\alpha}\zeta)^N}{\Gamma(2N/\alpha)} = \sqrt{\frac{N}{\alpha\pi}} \left(\frac{\alpha e}{2N}\right)^{2N/\alpha} \left(N^{\frac{2\delta}{\alpha}}\zeta\right)^N (1 + O(1/N)) = O(N^{-k})$$

holds for any power k of $1/N$, in view of $2N(1 - \delta)/\alpha > 0$. Since

$$g_\zeta(1) = \frac{N^{2\delta/\alpha}\zeta}{\Gamma(2/\alpha)}$$

we conclude, by (4.11),

$$r_1 = O(N^{2\delta/\alpha}). \quad (4.13)$$

According to the second mean value theorem (see e.g. [15]), for each $j \in \{0, \dots, N-1\}$ there exist $x_j \in [0, 1]$ such that

$$\int_0^1 \left(\frac{1}{2} - t\right) dg_\zeta(j+t) = -\frac{1}{2}(g_\zeta(j+x_j) - g_\zeta(j)) + \frac{1}{2}(g_\zeta(j+1) - g_\zeta(j+x_j))$$

Taking this into consideration, (4.12) can thus be written as

$$r_2 = \frac{1}{2} \sum_{j=1}^N (2g_\zeta(j+x_j) - (g_\zeta(j) + g_\zeta(j+1))) . \quad (4.14)$$

Some considerations about (4.14) are required. We have to avoid to take absolute value inside the sum since any estimate that disregards the change of sign in (4.14), leads r_2 to be of the leading order of the integral (4.8) given by $O\left(N^\delta e^{N^\delta |\zeta|^{\alpha/2}}\right)^4$. This follows by (3.9) and the fact that there are $O(N^{\delta/2})$ terms contributing to the sum (4.14), in view of Lemma 3.5. One needs to be careful and exploit the change of sign in a clever way in order to reduce the dependence on N from the number of terms of this sum. Because the estimates involve exponential growth, it is convenient to divide r_2 by the maximum value of $N^{\delta/2}g_{|\zeta|}(x)$ (see (3.9)). We set

$$\hat{r}_i = \frac{r_i}{N^{\delta/2}g_{|\zeta|}(x^*)} \quad (4.15)$$

for $i = 1, 2$, and note by (4.13) that \hat{r}_1 is exponentially small in N^δ .

Writing $\zeta = |\zeta| e^{i\theta/N^\beta}$ with $\theta \in \mathbb{R}$, we have by definition (3.7)

$$g_\zeta(x) = g_{|\zeta|}(x) \cos \theta N^{-\beta} x + i g_{|\zeta|}(x) \sin \theta N^{-\beta} x. \quad (4.16)$$

As r_2 is a linear function of g_ζ , it suffices to estimate its real part $\Re(r_2)$, since the estimate of $\Im(r_2)$ can be done in analogous manner.

⁴We have $g_{|\zeta|}(x^*) = O\left(N^{\delta/2} e^{N^\delta |\zeta|^{\alpha/2}}\right)$, the Euler–Maclaurin formula (4.10) gives an extra N^δ and $N^{-\delta/2}$ results from the Gaussian integration in the steepest–descent method. See Part **II**. for more detail.

The estimation of the real and imaginary parts of (4.16) depends on the period

$$p = \frac{2\pi}{|\theta|} N^\beta \quad (4.17)$$

of oscillation of $g_\zeta(x)$. For this, let $n_N(\theta)$ be the cardinality of the set

$$A_N(\theta) = \left\{ l \in \mathbb{N} : \frac{|\theta|}{\pi} N^{-\beta} < l \leq \frac{|\theta|}{\pi} N^{1-\beta} \right\}. \quad (4.18)$$

The number $n_N(\theta)$ counts how many oscillations between the maximum and minimum value of $\cos \theta N^{-\beta} x$ there are as x varies in the interval $[1, N]$. For pedagogical reason, we divide the estimate in two cases (i) $n_N(\theta) = O(1)$ and (ii) $n_N(\theta) = O(N^\varepsilon)$ for some $0 < \varepsilon \leq 1 - \beta$ ⁵. The estimate for the first case can be done with less effort. In the second case, which may also include the previous one, the estimate is more subtle and leads to sharper result.

(i) If $n_N(\theta) = n = O(1)$, we write (4.14) as

$$r_2 = r_2^{(1)} + r_2^{(2)}$$

where the real part of $r_2^{(i)}$, with $i = 1, 2$, is given by

$$\Re r_2^{(i)} = \sum_{j \in A_N^{(i)}} \left(\Re g_\zeta(j + x_j) - \left(\frac{\Re g_\zeta(j) + \Re g_\zeta(j+1)}{2} \right) \right) \quad (4.19)$$

with $A_N^{(i)}$ being the set of points $j \in \{1, \dots, N\}$ such that

$$\Re g_\zeta(j+1) - \Re g_\zeta(j + x_j) \begin{cases} \geq 0 & \text{if } i = 1 \\ < 0 & \text{if } i = 2 \end{cases}.$$

Let $(j_k)_{k=1}^L$ denote a sequence of points right before $\Re g_\zeta(j+1) - \Re g_\zeta(j + x_j)$, as a function of $j \in \{1, \dots, N\}$, changes its sign:

$$\begin{aligned} A_N^{(1)} &= \{1, \dots, j_1\} \cup \{j_2 + 1, \dots, j_3\} \cup \dots \cup \{j_{L-1} + 1, \dots, j_L\} \\ A_N^{(2)} &= \{j_1 + 1, \dots, j_2\} \cup \{j_3 + 1, \dots, j_4\} \cup \dots \cup \{j_L + 1, \dots, N\}. \end{aligned}$$

Since $0 \leq x_j \leq 1$ and $g_{|\zeta|}(x)$ is increasing in $[1, x^*)$ and decreasing in $(x^*, N]$, the points $(j_k)_{k=1}^L$ are essentially determined by the oscillations of the function $\cos \theta N^{-\beta} x$ in $\Re g_\zeta(x) = g_{|\zeta|}(x) \cos \theta N^{-\beta} x$ and $L = O(n_N(\theta)) = O(1)$, by hypothesis.

By definition, we have

$$\begin{aligned} \left| \Re r_2^{(1)} \right| &\leq \frac{1}{2} \left| \sum_{j \in A_N^{(1)}} (\Re g_\zeta(j+1) - \Re g_\zeta(j)) \right| \\ &= \frac{1}{2} |\Re g_\zeta(j_1 + 1) - \Re g_\zeta(1) + \dots + \Re g_\zeta(j_L + 1) - \Re g_\zeta(j_{L-1} + 1)| \end{aligned}$$

⁵We set $\varepsilon = 0$ when $\beta \geq 1$. In this case $n_N(\theta)$ is always $O(1)$. If $\beta < 1$, $n_N(\theta) = O(1)$ when $\theta = O(N^{-1+\beta})$.

and

$$\begin{aligned} |\Re r_2^{(2)}| &< \frac{1}{2} \left| \sum_{j \in A_N^{(2)}} (\Re(g_\zeta(j)) - \Re(g_\zeta(j+1))) \right| \\ &= \frac{1}{2} |\Re g_\zeta(j_1+1) - \Re g_\zeta(j_2+1) + \cdots + \Re g_\zeta(j_L+1) - \Re g_\zeta(N+1)| \end{aligned}$$

so that

$$|\Re r_2| \leq \sum_{k=1}^L g_{|\zeta|}(j_k+1) + \frac{g_{|\zeta|}(1) + g_{|\zeta|}(N+1)}{2}$$

yields, together with (4.15), (4.13), Lemma 3.3 and the fact that the same holds for $\Im m(r_2)$,

$$|\hat{r}_2| \leq O\left(\frac{1}{N^{\delta/2}}\right).$$

(ii) Let $n_N(\theta) = O(N^\varepsilon)$ for some $0 < \varepsilon \leq 1 - \beta$. Integrating (4.12) by parts gives

$$\begin{aligned} r_2 &= \sum_{j=1}^N \int_0^1 \left(\frac{1}{2} - t\right) dg_\zeta(j+t) \\ &= \sum_{j=1}^N \left(\left(\frac{1}{2} - t\right) g_\zeta(j+t) \Big|_0^1 + \int_0^1 g_\zeta(j+t) dt \right) \\ &= \sum_{j=1}^N \left(\int_0^1 g_\zeta(j+t) dt - \frac{1}{2} (g_\zeta(j) + g_\zeta(j+1)) \right). \end{aligned} \quad (4.20)$$

We now split the above sum into

$$r_2 = r_2^\sqcup + r_2^\sqcap \quad (4.21)$$

where the real part of $r_2^{\sqcup(\cap)}$ is given by

$$\Re r_2^{\sqcup(\cap)} = \sum_{j \in A_N^{\sqcup(\cap)}} \left(\int_0^1 \Re g_\zeta(j+t) dt - \frac{1}{2} (\Re g_\zeta(j) + \Re g_\zeta(j+1)) \right)$$

with $A_N^{\sqcup(\cap)}$ being the set of points $j \in \{1, \dots, N\}$ such that $(\Re g_\zeta)''(j) \geq 0$ (< 0).

Let us note that the function $\Re g_\zeta(x) = g_{|\zeta|}(x) \cos \theta x$ always has a well defined concavity and the cardinality of inflection points is of same order in N of the cardinality of critical points, since the main function responsible for both, the number of oscillations and changes of concavity, is the cosine.

Let $(k_i)_{i=1}^L$ denote a sequence of points in $\{1, \dots, N\}$ right before $(\Re g_\zeta)''(j)$ changes sign. Analogously, we have

$$\begin{aligned} A_N^\sqcap &= \{1, \dots, k_1\} \cup \{k_2+1, \dots, k_3\} \cup \cdots \cup \{k_{L-1}+1, \dots, k_L\} \\ A_N^\sqcup &= \{k_1+1, \dots, k_2\} \cup \{k_3+1, \dots, k_4\} \cup \cdots \cup \{k_L+1, \dots, N\} \end{aligned}$$

where, by the same reason as in item (i), $L = O(n_N(\theta)) = O(N^\varepsilon)$ and, consequently,

$$k_{i+1} - k_i = O(N^{1-\varepsilon}) \quad (4.22)$$

holds for $i = 1, \dots, L-1$. Note also that, by (4.17),

$$\theta = O(N^{\varepsilon+\beta-1}) \quad (4.23)$$

Applying Lemma 3.6 (and Corollary 3.7) to each interval $I_i = \{k_i + 1, \dots, k_{i+1}\}$, $i = 0, \dots, L$ ($k_0 \equiv 0$ and $k_{L+1} = N$) of size $K = O(N^{1-\varepsilon})$ with $f(x)$ replaced by $\Re g_\zeta(x)$ and $\Delta = 1$, yields

$$|\Re r_2| \leq \sum_{i=1}^L \left| \Re g_\zeta(t_i) - \frac{\Re g_\zeta(k_i) + \Re g_\zeta(k_i + 1)}{2} \right| + |\Re g_\zeta(1)| + |\Re g_\zeta(N+1)|, \quad (4.24)$$

with t_k defined by the mean value theorem $\Re g_\zeta(t_i) = \int_{k_i}^{k_{i+1}} \Re g_\zeta(x) dx$. Note that the points $(k_i)_{i=1}^L$ are closed to the inflection points $(x_i)_{i=1}^L$ of $\Re g_\zeta(x)$ and, moreover, the value of $\Re g_\zeta(x)$ at these points are small compared with the maximum value $g_{|\zeta|}(x^*)$. We shall estimate the order of $\Re g_\zeta(x_i)$ and use Lemma 3.5 to reduce the number of terms involved in the sum (4.21).

Taking the second derivative of the real part of (4.16), we obtain

$$(\Re g)''(x) = (g_{|\zeta|}''(x) - \theta^2 N^{-2\beta} g_{|\zeta|}(x)) \cos \theta N^{-\beta} x - 2\theta N^{-\beta} g_{|\zeta|}'(x) \sin \theta N^{-\beta} x$$

Since derivatives of $g_{|\zeta|}(x)$ increases its value by a logarithm of N factor (see equation (3.11)), combined with (4.23), it gives

$$\frac{(\Re g)''(x)}{g_{|\zeta|}(x)} = (O(\log^2 N) + O(N^{2(\varepsilon-1)})) \cos \theta N^{-\beta} x + O(N^{\varepsilon-1} \log N) \sin \theta N^{-\beta} x. \quad (4.25)$$

But we have, on the other hand,

$$(\Re g)''(x_i) = (g_{|\zeta|}''(x_i) - \theta^2 N^{-2\beta} g_{|\zeta|}(x_i)) \cos \theta N^{-\beta} x_i - 2\theta N^{-\beta} g_{|\zeta|}'(x_i) \sin \theta N^{-\beta} x_i = 0$$

holds at each inflection point x_i . This together with (4.25) implies that the inflection point x_i must be at $O(1/\log N)$ distance from the k -th zero of $\cos \theta N^{-\beta} x$. Indeed, defining $\Delta_i = O(1/\log N)$ by

$$x_i = \frac{(2i-1)\pi}{2|\theta| N^{-\beta}} + \Delta_i$$

we have

$$\begin{aligned} \cos \theta N^{-\beta} x_i &= \cos(\pm(i-1/2)\pi + \theta N^{-\beta} \Delta_i) = \pm(-1)^i \sin \theta N^{-\beta} \Delta_i = O(N^{\varepsilon-1}/\log N) \\ \sin \theta N^{-\beta} x_i &= \sin(\pm(i-1/2)\pi + \theta N^{-\beta} \Delta_i) = \mp(-1)^i \cos \theta N^{-\beta} \Delta_i = O(1) \end{aligned}$$

and, together with (4.25), one sees that $(\Re g)''(x_i) = 0$ holds in the leading order. Since the points k_i , t_i and $k_i + 1$ are not distant from the inflection point x_i ($g_\zeta(x)$ varies slowly for each interval $k_i \leq x \leq k_i + 1$),

$$\frac{|\Re g_\zeta(x)|}{g_{|\zeta|}(x)} = |\cos \theta N^{-\beta} x| \leq O(N^{-1+\varepsilon}/\log N) \quad (4.26)$$

holds for x at the values $\{k_i, t_i, k_i + 1\}_{i=1}^L$.

The number of terms that contributes to (4.21), as well as to the sum (4.24), can be estimated using Lemma 3.5. Instead of an interval I of size N we shall consider an interval I' containing x^* with $O(N^{\delta/2} \log N)$ points. By (4.22), a number of order $N^{\delta/2} \log N / N^{1-\varepsilon}$ of terms give appreciably contribution to (4.24) and, together with (4.26), the fact that the same estimate holds for $\Im m g_\zeta(x)$ and (4.15), we conclude

$$|\hat{r}_2| = O(N^{-2(1-\varepsilon)})$$

uniformly in every closed interval of $0 < \varepsilon \leq 1 - \beta$.

II. The Method of Steepest Descents Equation (4.10) can be written as

$$S_N = N^{\delta/2} g_{|\zeta|}(x^*) \left(N^{\delta/2} \int_{N^{-\delta}}^{N^{1-\delta}} f(y) dy + \hat{r}_1 + \hat{r}_2 \right) \quad (4.27)$$

where, by the Stirling formula (see (3.14)),

$$f(y) = \frac{g_\zeta(N^\delta y)}{g_{|\zeta|}(x^*)} = \sqrt{\frac{2y}{\alpha |\zeta|^{\alpha/2}}} e^{N^\delta h(y)} (1 + O(1/N^\delta)) \quad (4.28)$$

with

$$h(y) = \frac{2y}{\alpha} \log \frac{\alpha e \zeta^{\alpha/2}}{2y} - |\zeta|^{\alpha/2} \quad (4.29)$$

Note that $\Re h(y) \leq 0$ holds for all $y > 0$ and attains to its maximum $\Re h(y^*) = 0$ at $y^* = \alpha |\zeta|^{\alpha/2} / 2$ inside the domain of integration $[N^{-\delta}, N^{1-\delta}]$, by condition $0 < \delta < 1$ and N large enough.

We now use the steepest descents technique to estimate the integral that appears in (4.27). This technique uses the Cauchy theorem to deform the interval of integration $[N^{-\delta}, N^{1-\delta}]$ into a curve \mathcal{C} :

$$I = \sqrt{\frac{2N^\delta}{\alpha |\zeta|^{\alpha/2}}} \int_{N^{-\delta}}^{N^{1-\delta}} \sqrt{y} e^{N^\delta h(y)} dy = \sqrt{\frac{2N^\delta}{\alpha |\zeta|^{\alpha/2}}} \int_{\mathcal{C}} \sqrt{\eta} e^{N^\delta h(\eta)} d\eta \quad (4.30)$$

where $h : \mathbb{C} \rightarrow \mathbb{C}$ is extended analytically to the complex plane, $\eta = y + iw$ and \mathcal{C} is a smooth curve with extreme points $\eta_1 = N^{-\delta}$ and $\eta_2 = N^{1-\delta}$ chosen in such a way that (a) it passes by the saddle point $\eta_0 = \alpha \zeta^{\alpha/2} / 2$ ($|\eta_0| = y^*$) defined implicitly by

$$h'(\eta_0) = \frac{2}{\alpha} \log \frac{\alpha \zeta^{\alpha/2}}{2\eta_0} = 0 \quad (4.31)$$

and (b) it maximizes the function $\Re h(y, w)$ along a level curve

$$\Im h(y, w) = c$$

in a neighborhood U_0 of η_0 . If, in addition,

$$\Re h(y, w) \geq \max\{\Re h(N^{-\delta}, 0), \Re h(N^{1-\delta}, 0)\} \quad (4.32)$$

holds along \mathcal{C} , then the main contribution to (4.30) will be given by the saddle point η_0 ; if, on the other hand, (4.32) cannot be satisfied to any such curve \mathcal{C} , the main contribution to the integral (4.30) will be given by the extreme points.

At the extreme points, neither η_1 nor η_2 plays an important role, since both leave the integral (4.30) exponentially small with N . So, the contribution to (4.30) is given by the vicinity of the saddle point.

Expanding h in Taylor series about $\eta_0 = \alpha \zeta^{\alpha/2}/2 = \alpha |\zeta|^{\alpha/2} e^{i\alpha N^{-\beta}\theta/2}/2$, gives

$$\begin{aligned} h(\eta) &= h(\eta_0) + \frac{1}{2}h''(\eta_0)(\eta - \eta_0)^2 + O((\eta - \eta_0)^3) \\ &= \zeta^{\alpha/2} - |\zeta|^{\alpha/2} - \frac{2}{\alpha^2 |\zeta|^{\alpha/2}} \rho^2 e^{i(2\varphi - \alpha\theta N^{-\beta}/2)} + O((\eta - \eta_0)^3) \end{aligned}$$

with $\eta - \eta_0 = \rho e^{i\varphi} \in U_0$. We choose \mathcal{C} so that $2\varphi - \alpha\theta N^{-\beta}/2 = 0$ at the the saddle point. Applying the steepest descents technique, the integral (4.30) can be approximate by a Gaussian integral in the vicinity U_0 of η_0 , resulting (see e.g. [14], for details)

$$\begin{aligned} S_N &= N^{\delta/2} g_{|\zeta|}(x^*) \left(e^{N^\delta(\zeta^{\alpha/2} - |\zeta|^{\alpha/2})} \sqrt{\frac{2N^\delta}{\alpha |\zeta|^{\alpha/2}}} \sqrt{\frac{2\pi\eta_0}{-N^\delta h''(\eta_0)}} \left(1 + O\left(\frac{1}{N^\delta}\right) \right) + \hat{r}_1 + \hat{r}_2 \right) \\ &= N^{\delta/2} g_{|\zeta|}(x^*) \left(e^{N^\delta(\zeta^{\alpha/2} - |\zeta|^{\alpha/2})} \sqrt{\frac{2\pi}{|\zeta|^{\alpha/2} \eta_0}} \left(1 + O\left(\frac{1}{N^\delta}\right) \right) + \hat{r}_1 + \hat{r}_2 \right) \end{aligned} \quad (4.33)$$

Now, since by (3.8) $\alpha N^\delta |\zeta|^{\alpha/2}/2 < N$,

$$\begin{aligned} \left| \exp\left(N^\delta(\zeta^{\alpha/2} - |\zeta|^{\alpha/2})\right) \right| &= \exp\left(N^\delta |\zeta|^{\alpha/2} (\cos \alpha N^{-\beta}\theta/2 - 1)\right) \\ &\geq \exp(-\alpha\theta^2 N^{1-2\beta}) \end{aligned} \quad (4.34)$$

and, provided $\beta \geq 1/2$, it follows from the estimates of r_1 and r_2 in **I**. that

$$S_N = \frac{\alpha}{2} \zeta^{\alpha/2} N^\delta \exp\left(\zeta^{\frac{\alpha}{2}} N^\delta\right) (1 + E_N^{\alpha,\delta}(\zeta))$$

with

$$|E_N^{\alpha,\delta}(\zeta)| \leq O(N^{-\delta/2})$$

whenever $\zeta \in S(\theta N^{-1/2}, K^{\alpha,\delta})$, where $K^{\alpha,\delta} = (2N^{1-\delta}/\alpha)^{2/\alpha}$. Therefore, we obtain from (4.3)

$$\frac{1}{N^{\delta+2\gamma}} K_N^\alpha \left(\frac{Z}{N^\gamma}, \frac{W}{N^\gamma} \right) = \frac{\alpha^2}{4\pi} (Z\bar{W})^{\frac{\alpha}{2}-1} e^{N^\delta \left((Z\bar{W})^{\frac{\alpha}{2}} - \frac{|Z|^\alpha}{2} - \frac{|W|^\alpha}{2} \right)} (1 + E_N^{\alpha,\delta}(Z\bar{W})) \quad (4.35)$$

where we have used (4.5) with $0 < \delta < 1$. In particular, taking $\delta \nearrow 1$,

$$\frac{1}{N} K_N^\alpha(Z, W) = \frac{\alpha^2}{4\pi} (Z\bar{W})^{\frac{\alpha}{2}-1} e^{N \left((Z\bar{W})^{\frac{\alpha}{2}} - \frac{|Z|^\alpha}{2} - \frac{|W|^\alpha}{2} \right)} (1 + E_N^{\alpha,1}(Z\bar{W})). \quad (4.36)$$

□

Remark 4.1 Equation (4.34) prevents $\zeta = Z\bar{W}$ to be defined in a sector $S(\theta N^{-\beta}, K^{\alpha,\delta})$ of opening wider than $O(N^{-1/2})$. The introduction of the scale $\delta < 1$ guarantees that the main contribution to (4.30) comes from the saddle point for any $\zeta \in \mathbb{C}$ fixed. Note that $K^{\alpha,\delta} = O(N^{2(1-\delta)/\alpha})$ and for $\delta = 1$ we need $|\zeta| \leq K^{\alpha,1} = (2/\alpha)^{2/\alpha}$ (see Remark 3.4). As the calculation in the appendix below indicates, $|\zeta|$ may be even smaller than that, depending on the sector opening τ .

A Taylor Remainder

Let $f_N(\zeta) = N\zeta e^{N\zeta}$ be a function defined for $\zeta = |\zeta| e^{i\theta} \in \mathbb{C}$ and N a fixed natural number. Its Taylor remainder with respect to the polynomial $S_N(\zeta) = N\zeta + \cdots + \frac{1}{(N-1)!}(N\zeta)^N$ of order N can be expressed by the Lagrange formula (see e.g. [19])

$$R_N(\zeta) = f_N(\zeta) - S_N(\zeta) = \frac{1}{(N+1)!} g_N^{(N+1)}(a)$$

for some $0 < a < 1$, where $g_N(x) = f_N(x\zeta)$, $x \in [0, 1]$, satisfies

$$g_N^{(r)}(x) = (rN^r + N^{r+1}x\zeta) \zeta^r e^{Nx\zeta} \quad (\text{A.1})$$

for every $r \in \mathbb{N}$, by induction.

Writing

$$S_N(\zeta) = f_N(\zeta)(1 + E_N(\zeta))$$

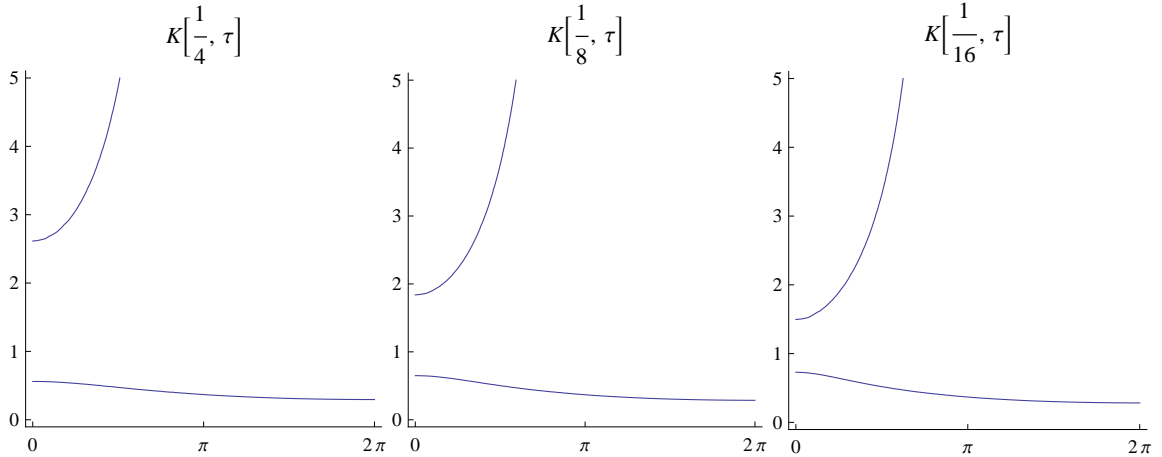
the error function $E_N(\zeta) = R_N(\zeta)/f_N(\zeta)$ is estimated for ζ in a sectorial domain $S(\tau, K) = \{\zeta \in \mathbb{C} : |\arg(\zeta)| < \tau/2, |\zeta| < K\}$ using (A.1) together with the Stirling formula $r! = \sqrt{2\pi r}(r/e)^r(1 + O(1/r))$:

$$|E_N(\zeta)| = \frac{1}{\sqrt{2\pi N}} |1 + a\zeta| e^N |\zeta|^N e^{-N(1-a)|\zeta|\cos\theta} (1 + O(1/N))$$

so $\sup_{\zeta \in S(\tau, K)} |E_N(\zeta)| = O(1/\sqrt{N})$ where $K = K(a, \tau) > 0$ is given by the smallest solutions of

$$K e^{-(1-a)K \cos \tau/2 + 1} = 1, \quad (\text{A.2})$$

which exists and is continuous for all $0 < a < 1$ and $\tau \in [0, 2\pi]$. The implicit solutions of (A.2) for $K = K(a, \tau)$ are described in figure below for $a = 1/2, 1/8$ and $1/16$.



B Proof of Corollary 1.6

Assuming temporarily that (1.10) holds with $Z = W = r$, we observe that by (1.11)

$$\begin{aligned} Z_i &= r + \frac{1}{\sqrt{N}} \frac{2z_i}{\alpha |r|^{\alpha/2-1}} + O(1/N) \\ &= r \exp \left(\frac{1}{\sqrt{N}} \frac{2z_i}{\alpha |r|^{\alpha/2-1}} + O(1/N) \right) \end{aligned}$$

and

$$\arg(Z_i \bar{Z}_j) < \theta / \sqrt{N},$$

for some $\theta > 0$ and any i, j , if N is large enough, say $N > N_1$. We take, in addition, $N > N_0$ where N_0 is given by (3.8) with $1/|\zeta|$ and $|\zeta|$ replaced by $1/\min_{i,j} (|Z_i \bar{Z}_j|)$ and $\max_{i,j} (|Z_i \bar{Z}_j|)$, respectively. So, for $N > \max(N_0, N_1)$ equation (1.10) holds with (r, r) and (Z_i, Z_j) , for any i, j , in the place of (Z, W) . From equation (1.9) and (1.11), it holds for $r \in \mathbb{C}$ with $0 < |r| < (2/\alpha)^{1/\alpha}$, whose closure is the support of the eigenvalues density (see eq. 2.6).

Now, applying the Taylor expansion

$$(1+w)^{\alpha/2} = 1 + \frac{\alpha}{2}w + \frac{\alpha}{4} \left(\frac{\alpha}{2} - 1 \right) w^2 + O(w^3)$$

to the exponent of $K_N^\alpha(Z_i, Z_j)$, yields

$$N \left((Z_i \bar{Z}_j)^{\alpha/2} - \frac{1}{2} |Z_i|^\alpha - \frac{1}{2} |Z_j|^\alpha \right) = A_{ij} + i\sqrt{N} B_{ij} + O(1/\sqrt{N}) \quad (\text{B.1})$$

where

$$\begin{aligned} A_{ij} &= z_i \bar{z}_j - \frac{1}{2} |z_i|^2 - \frac{1}{2} |z_j|^2, \\ B_{ij} &= \lambda_i - \lambda_j \end{aligned}$$

and

$$\lambda_i = |r|^{\alpha/2+1} \Im \frac{z_i}{r} + \frac{1}{2\sqrt{N}} |r|^2 \left(1 - \frac{2}{\alpha} \right) \Im \frac{z_i^2}{r^2}$$

is a real number. Let C_N and D_N denote $n \times n$ matrices with respective entries $(C_N)_{ij} = \frac{1}{\pi} \exp \left(A_{ij} + i\sqrt{N} B_{ij} \right) (1 + O(1/\sqrt{N}))$ and $D_{ij} = \frac{1}{\pi} \exp(A_{ij}) (1 + O(1/\sqrt{N}))$ ($= C_{ij}$ with $B_{ij} = 0$). If we write $\Lambda_N = \text{diag} \left(\exp(i\sqrt{N} \lambda_i) \right)$, then $C_N = \Lambda_N D_N \bar{\Lambda}_N$, $\Lambda_N \bar{\Lambda}_N = I$ ($\bar{\Lambda}_N$ and I are the complex conjugate of Λ_N and the identity matrix) and

$$\det C_N = \det \Lambda_N D_N \bar{\Lambda}_N = \det D_N \bar{\Lambda}_N \Lambda_N = \det D_N.$$

by Cauchy-Binet formula. This concludes the proof since, by (1.6) (1.10) and (B.1), the l.h.s of (1.12) is the determinant of a matrix whose asymptotic expansion is given by C_N and

$$\lim_{N \rightarrow \infty} \det C_N = \lim_{N \rightarrow \infty} \det D_N = \det (\mathbb{K}(z_i, z_j))_{i,j=1}^n$$

by continuity. □

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